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Six-Dimensional Dirac Equation

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The Dirac equation in six-dimensional relativity (three space and three time) is considered and shown to correspond to particles which have spatial spin-$\frac{1}{2}$ and temporal spin-$\frac{1}{2}$. Explicit forms of the spinor transformation are found. Plane wave solutions are obtained and their properties are given in terms of spatial and temporal spins and helicities. An expression is found for the charge conjugation operator.

1. INTRODUCTION

When certain schemes are introduced to extend the four-dimensional Lorentz transformations to deal with superluminal transformations between inertial frames it is found that some space-time coordinates become imaginary (Recami and Mignani, 1974; Maccarrone *et al.*, 1983). But it has been shown (Cole, 1977) that the coordinates remain real and the transformations linear when the number of coordinates is raised to six. This introduces two extra time coordinates. Extra time coordinates have been introduced by various authors (Dattoli and Mignani, 1978; Demers, 1975; Pappas, 1982; Patty, 1982; Patty and Smalley, 1985; Pavsic, 1981; Vysin, 1978; Ziino, 1981) and criticisms of this approach have been made (Strnad, 1983; Weinberg, 1980; Spinelli, 1979; Ray, 1979; Dorling, 1970). Since there are different ways of introducing extra coordinates, the methods of the above authors give differing predictions to some degree. For example, the methods of Pappas and Ziino predict no transverse Doppler effect and no Thomas precession.

Any reasonable theory involving six dimensions must (i) include the standard four-dimensional theory as a special case, (ii) explain why we detect only one time dimension, and (iii) make testable predictions which are not made by the standard theory. The model of Cole (1980) satisfies requirement

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(i). Requirement (ii) is demonstrated by showing that (Cole and Buchanan, 1982) the energy magnitude required to turn the time vector of a particle of mass M through an angle ϑ is $2Mc^2 \sin(\vartheta/2)$, which for $M = 1$ kg and $\vartheta = 1°$ is 1.57×10^{15} J. Thus an everyday macroscopic object requires a very large energy input in order to turn its time vector by an appreciable amount, and thus deviations from a common time direction are not observed. An attempt at requirement (iii) has been made (Cole and Starr, 1985, 1990) by showing that in this model redshifts can be obtained with smaller observed velocity than is the case in the standard four-dimensional theory.

The work contained in this present paper was prompted by that of Patty and Smalley (1985), who introduced the Dirac equation in six dimensions and gave their interpretation of its consequences. In this paper we retain their representation of the γ matrices to study spinor transformations, the plane wave solution, and charge conjugation. It will emerge that the six-dimensional Dirac equation corresponds to particles which have spatial spin-$\frac{1}{2}$ and temporal spin-$\frac{1}{2}$.

The remainder of this section reproduces results already derived for use in later sections. Let I_n be the $n \times n$ identity matrix and let $g^{\mu\nu} = (-1, -1, -1, 1, 1, 1)$. The six-dimensional Dirac equation is

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0 \tag{1.1}$$

where the last three components of the 6-vector x represent time, ψ is an 8-component quantity, and the γ^μ ($\mu = 1, \ldots, 6$) are 8×8 matrices which satisfy

$$\begin{gathered}\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I_8 \\ \gamma^{i\dagger} = -\gamma^i, \qquad (\gamma^i)^2 = -I_8 \qquad (i = 1, 2, 3) \\ \gamma^{j\dagger} = \gamma^j, \qquad (\gamma^j)^2 = I_8 \qquad (j = 4, 5, 6)\end{gathered} \tag{1.2}$$

The representation of Patty and Smalley to be used is

$$\gamma^j = \begin{pmatrix} 0 & 0 & 0 & \sigma_j \\ 0 & 0 & \sigma_j & 0 \\ 0 & -\sigma_j & 0 & 0 \\ -\sigma_j & 0 & 0 & 0 \end{pmatrix} \qquad (j = 1, 2, 3)$$

$$\gamma^4 = \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & -I_2 & 0 \\ 0 & 0 & 0 & -I_2 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} 0 & 0 & -iI_2 & 0 \\ 0 & 0 & 0 & iI_2 \\ iI_2 & 0 & 0 & 0 \\ 0 & -iI_2 & 0 & 0 \end{pmatrix}$$

$$\gamma^6 = \begin{pmatrix} 0 & 0 & 0 & I_2 \\ 0 & 0 & -I_2 & 0 \\ 0 & -I_2 & 0 & 0 \\ I_2 & 0 & 0 & 0 \end{pmatrix}$$

An important quantity arising later is the 8×8 matrix

$$A \equiv i\gamma^4\gamma^5\gamma^6 = \begin{pmatrix} 0 & -I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & I_2 & 0 \end{pmatrix} \tag{1.3}$$

which has the properties $A = A^\dagger$, $A^2 = I$, and $\gamma^{\mu\dagger} A = A\gamma^\mu$ ($\mu = 1, \ldots, 6$). A Lorentz transformation linking the spacetime coordinates x^μ and $x^{\mu'}$ associated with two inertial frames F and F' is of the form

$$x^{\nu'} = a^\nu{}_\mu x^\mu$$

where

$$a_\mu{}^\nu a^\mu{}_\sigma = \delta^\nu{}_\sigma \qquad \text{and} \qquad \frac{\partial}{\partial x^\mu} = a^\nu{}_\mu \frac{\partial}{\partial x^{\nu'}} \tag{1.4}$$

The way that such a transformation is constructed is given as follows (Cole, 1980). The motion of a particle in an inertial frame is specified by the unit vector $\boldsymbol{\alpha}$ along the projection of its path in the time subspace and by its velocity $\mathbf{v} = d\mathbf{r}/dt$, where dt is measured along the projection in the direction of $\boldsymbol{\alpha}$. The energy of the particle is $\mathbf{E} = E\boldsymbol{\alpha}$, that is, directed along $\boldsymbol{\alpha}$. Writing $\gamma = (1 - v^2)^{-1/2}$, then the 6-momentum of a particle is

$$P^\mu = m\gamma \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ \mathbf{E} \end{pmatrix}$$

where m is its rest mass, $\mathbf{p}$ is its 3-momentum, and units in which $c = 1$ are used. The transformation between two inertial frames is given by specifying in each frame the quantities $\mathbf{v}$ and $\boldsymbol{\alpha}$ for the two spatial origins O and O' of F and F'. The simplest nontrivial transformation corresponding to the simplicity of the standard configuration of the four-dimensional theory is given when the x and x', y and y', and z and z' axes are parallel and motion with relative speed v is along the x–x' directions. The time vectors of O and O' are along the t_1 and t_1' axes, respectively, the t_3 and t_3' axes are parallel, and the angle between the t_1 and t_1' axes in F is θ. Then it has been shown (Cole, 1985) that

$$(a^\mu{}_\nu)=\begin{pmatrix} \dfrac{\gamma(\gamma+\cos\theta)}{1+\gamma\cos\theta} & 0 & 0 & -\gamma v & \dfrac{-v\gamma^2\sin\theta}{1+\gamma\cos\theta} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\gamma v & 0 & 0 & \gamma\cos\theta & \gamma\sin\theta & 0 \\ \dfrac{v\gamma^2\sin\theta}{1+\gamma\cos\theta} & 0 & 0 & -\gamma\sin\theta & 1-\dfrac{\gamma^2\sin^2\theta}{1+\gamma\cos\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.5}$$

which reduces in the standard limit $\theta\to 0$ to

$$(a^\mu{}_\nu)=\begin{pmatrix} \gamma & 0 & 0 & -\gamma v & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\gamma v & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The results that follow are new. In Section 2 we introduce the spinor transformation $\psi'(x')=S(a)\psi(x)$ and obtain an explicit form of $S(a)$ for the transformation (1.5). The 6-current is introduced as $\psi^\dagger A\gamma^\mu\psi$ and it is shown that the normalization of ψ using $\psi^\dagger A\psi$ will be frame invariant. In Section 3, the plane wave solutions are obtained and their properties are given in terms of the spatial and temporal spins and helicities. Charge conjugation is discussed in Section 4 and the charge conjugation operator is found to be $C=\gamma^1\gamma^3\gamma^4\gamma^6$.

2. SPINOR TRANSFORMATIONS

Under a Lorentz transformation (1.4), the spinor transforms as

$$\psi'(x')=\psi'(ax)=S(a)\psi(x)=S(a)\psi(a^{-1}x')$$

with $S(a^{-1})=S^{-1}(a)$. As in the standard four-dimensional case we may easily show that S must satisfy

$$S^{-1}\gamma^{\nu}S=a^{\nu}{}_{\mu}\gamma^{\mu} \tag{2.1}$$

and we may further show, with A defined in (1.3), that

$$AS^{\dagger}A=S^{-1} \tag{2.2}$$

To prove this last result, consider an infinitesimal transformation $a^{\nu}{}_{\mu}=\delta^{\nu}{}_{\mu}+\Delta\omega^{\nu}{}_{\mu}$, where $\Delta\omega_{\nu\mu}=-\Delta\omega_{\mu\nu}$. On expanding S in powers of $\Delta\omega_{\mu\nu}$ and keeping only the linear terms, one finds

$$\begin{aligned} S&=I_8-(i/4)\sigma^{\mu\nu}\Delta\omega_{\mu\nu}\\ S^{-1}&=I_8+(i/4)\sigma^{\mu\nu}\Delta\omega_{\mu\nu}\\ S^{\dagger}&=I_8+(i/4)\sigma^{\mu\nu\dagger}\Delta\omega_{\mu\nu}\end{aligned} \tag{2.3}$$

with

$$\sigma^{\mu\nu}=-\sigma^{\nu\mu}=(i/2)[\gamma^{\mu},\gamma^{\nu}] \tag{2.4}$$

Using (2.4) and (1.3), it is easily verified that $\sigma^{\mu\nu\dagger}=A\sigma^{\mu\nu}A$ and it follows from (2.3) that $AS^{\dagger}A=S^{-1}$. Building up the transformation from infinitesimals, the result (2.2) follows.

Taking the conjugate of (1.1) gives

$$-i\left(\frac{\partial\psi}{\partial x^{\mu}}\right)^{\dagger}\gamma^{\mu\dagger}-m\psi^{\dagger}=0$$

On defining

$$j^{\mu}\equiv\psi^{\dagger}A\gamma^{\mu}\psi \tag{2.5}$$

we may show that

$$\frac{\partial}{\partial x^{\mu}}j^{\mu}=0$$

and using (2.1) and (2.2), it follows that

$$j^{\mu'}(x')=a^{\mu'}{}_{\sigma}j^{\sigma}(x)$$

Thus the quantity j^{μ} has zero divergence and transforms as a 6-vector and may therefore be thought of as the 6-current.

Further,

$$\psi'^{\dagger}(x')A\psi'(x')=\psi^{\dagger}(x)S^{\dagger}AS\psi(x)=\psi^{\dagger}(x)A\psi(x)$$

and so a normalization of ψ using the quantity $\psi^{\dagger}A\psi$ will be frame invariant.

Note that in the corresponding results of the four-dimensional theory, the quantity A is replaced by the matrix γ^4 which accompanies the single time derivative in the Dirac equation.

We end this section by illustrating these results using the special form of the transformation (1.5). The infinitesimal version of (1.5) is

$$(a^\mu{}_\nu) = \begin{pmatrix} 1 & 0 & 0 & -\Delta\omega & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\Delta\omega & 0 & 0 & 1 & \Delta\phi & 0 \\ 0 & 0 & 0 & -\Delta\phi & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\equiv I_6 + \Delta\omega J + \Delta\phi K$$

where $J^1{}_4 = J^4{}_1 = -1$, $K^4{}_5 = -K^5{}_4 = 1$, and all other elements are zero. The complete transformation may then be built up as

$$x^{\nu'} = \lim_{N\to\infty} \left(I_6 + \frac{\omega}{N} J + \frac{\phi}{N} K\right)^\nu{}_{\alpha_1} \left(I_6 + \frac{\omega}{N} J + \frac{\phi}{N} K\right)^{\alpha_1}{}_{\alpha_2} \ldots x^{\alpha_N}$$

$$= \lim_{N\to\infty} \left[\left(I_6 + \frac{\omega}{N} J + \frac{\phi}{N} K\right)^N\right]^\nu{}_\mu x^\mu$$

$$= [\exp(\omega J + \phi K)]^\nu{}_\mu x^\mu = (\exp Q)^\nu{}_\mu x^\mu$$

where $Q \equiv \omega J + \phi K$. One may show that $Q^3 = (\omega^2 - \phi^2) Q$, so that any power of Q can be written in terms of I_6, Q, or Q^2. Expanding the exponentials, one finds

$$\exp(\omega J + \phi K) = I_6 + \frac{\sinh\sqrt{\omega^2 - \phi^2}}{\sqrt{\omega^2 - \phi^2}} Q + \frac{\cosh\sqrt{\omega^2 - \phi^2} - 1}{\omega^2 - \phi^2} Q^2$$

On writing down the matrix Q, evaluating Q^2, and making the substitution $\phi/\omega = (\sin\theta)/v$, $\cosh\sqrt{\omega^2 - \phi^2} = \gamma\cos\theta$, then the transformation (1.5) is recovered exactly.

The spinor transformation S may be constructed similarly. For the infinitesimal transformation it is

$$I_8 - \frac{i}{4}\sigma^{\mu\nu}\Delta\omega_{\mu\nu} = I_8 - \frac{i}{4}(\Delta\omega\sigma^{14} - \Delta\omega\sigma^{41} + \Delta\phi\sigma^{45} - \Delta\phi\sigma^{54})$$

$$= I_8 - \frac{i}{2}(\Delta\omega\sigma^{14} + \Delta\phi\sigma^{45})$$

Hence

$$S = \lim_{N\to\infty}\left[I_8 - \frac{i}{2}\left(\frac{\omega}{N}\sigma^{14} + \frac{\phi}{N}\sigma^{45}\right)\right]^N = \exp\left[-\frac{i}{2}(\omega\sigma^{14} + \phi\sigma^{45})\right]$$

where

$$\sigma^{14} = \frac{i}{2}[\gamma^1, \gamma^4] = i\gamma^1\gamma^4 = -i\begin{pmatrix} 0 & 0 & 0 & \sigma_1 \\ 0 & 0 & \sigma_1 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ \sigma_1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\sigma^{45} = \frac{i}{2}[\gamma^4, \gamma^5] = i\gamma^4\gamma^5 = \begin{pmatrix} 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & -I_2 \\ I_2 & 0 & 0 & 0 \\ 0 & -I_2 & 0 & 0 \end{pmatrix}$$

Since $\sigma^{14\dagger} = -\sigma^{14}$ and $\sigma^{45\dagger} = \sigma^{45}$ it follows that

$$S^{-1} = \exp\left[\frac{i}{2}(\omega\sigma^{14} + \phi\sigma^{45})\right], \qquad S^\dagger = \exp\left[\frac{i}{2}(-\omega\sigma^{14} + \phi\sigma^{45})\right]$$

and it is then easily verified that (2.2) is satisfied.

3. PLANE WAVE SOLUTIONS

A plane wave solution of (1.1) corresponding to a particle of 6-momentum P^μ has the form

$$\psi = \exp(-iP_\mu x^\mu)w$$

where

$$(\gamma^\mu P_\mu)w = mw \tag{3.1}$$

The 6-current (2.5) corresponding to such a solution is

$$\begin{aligned} j^\mu &= w^\dagger A\gamma^\mu w = (2m)^{-1}[w^\dagger(\gamma^{\lambda\dagger}P_\lambda)A\gamma^\mu w + w^\dagger A\gamma^\mu(\gamma^\lambda P_\lambda)w] \\ &= (2m)^{-1}w^\dagger A(\gamma^\lambda\gamma^\mu + \gamma^\mu\gamma^\lambda)P_\lambda w = (m)^{-1}P^\mu w^\dagger Aw \end{aligned} \tag{3.2}$$

For a given 6-momentum P^μ, (3.1) has four linearly independent solutions w. It is convenient to specify these in terms of eigenvalues of spatial and temporal spin. The *spatial spin* Σ (measured in units of $\hbar/2$) has components

$$(\Sigma_1, \Sigma_2, \Sigma_3) = (\sigma^{23}, \sigma^{31}, \sigma^{12}) = (i\gamma^2\gamma^3, i\gamma^3\gamma^1, i\gamma^1\gamma^2)$$

satisfying the Pauli relations

$$\Sigma_j \Sigma_k = \delta_{jk} I_8 + i\varepsilon_{jkl} \Sigma_l$$

and given explicitly by

$$\Sigma_k = \begin{pmatrix} \sigma_k & 0 & 0 & 0 \\ 0 & \sigma_k & 0 & 0 \\ 0 & 0 & \sigma_k & 0 \\ 0 & 0 & 0 & \sigma_k \end{pmatrix}, \qquad (k=1, 2, 3)$$

Similarly the *temporal spin* $\boldsymbol{\tau}$ has components

$$(\tau_1, \tau_2, \tau_3) = (-\sigma^{56}, -\sigma^{64}, -\sigma^{45}) = (-i\gamma^5\gamma^6, -i\gamma^6\gamma^4, -i\gamma^4\gamma^5)$$

commuting with the Σ_k, satisfying the Pauli relations

$$\tau_j \tau_k = \delta_{jk} I_8 + i\varepsilon_{jkl} \tau_l$$

and given by

$$\tau_k = -\gamma^{k+3} A = -A\gamma^{k+3}, \qquad (k=1, 2, 3)$$

Explicitly,

$$\tau_1 = \begin{pmatrix} 0 & I_2 & 0 & 0 \\ I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & I_2 & 0 \end{pmatrix}$$

$$\tau_2 = \begin{pmatrix} 0 & 0 & 0 & iI_2 \\ 0 & 0 & -iI_2 & 0 \\ 0 & iI_2 & 0 & 0 \\ -iI_2 & 0 & 0 & 0 \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} 0 & 0 & -I_2 & 0 \\ 0 & 0 & 0 & I_2 \\ -I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \end{pmatrix}$$

The components of $\boldsymbol{\Sigma}$ and $\boldsymbol{\tau}$ in the directions of the unit 3-vectors $\hat{\mathbf{p}} = |\mathbf{p}|^{-1}\mathbf{p}$ and $\hat{\mathbf{E}} = E^{-1}\mathbf{E}$, respectively, will be known as the *spatial helicity* $\boldsymbol{\Sigma} \cdot \hat{\mathbf{p}}$ and the *temporal helicity* $\boldsymbol{\tau} \cdot \hat{\mathbf{E}}$. It is easily verified that these two 8×8 matrices commute with each other and with the matrix $\gamma^\mu P_\mu$ on the left-hand side of (3.1). We can therefore construct solutions w of (3.1) which are also simultaneous eigenvectors of the two helicities. Note that (3.2) with

$\mu = 4, 5, 6$ gives $\mathrm{E}w^\dagger Aw/m = -w^\dagger \boldsymbol{\tau} w$, so that $w^\dagger(\boldsymbol{\tau}\cdot\hat{\mathbf{E}})w = -Ew^\dagger Aw/m$. Thus, if w is an eigenvector of temporal helicity, then $w^\dagger Aw$ has sign opposite to that of the temporal helicity of w.

We shall obtain solutions w_r of (3.1) for $r = 1, 2, 3, 4$ satisfying

$$\begin{aligned}(\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}})w_r &= (-1)^{r-1}w_r\\ (\boldsymbol{\tau}\cdot\hat{\mathbf{E}})w_r &= (-1)^{r(r+1)/2}w_r\\ w_r^\dagger Aw_s &= -(-1)^{r(r+1)/2}\delta_{rs} \qquad (r, s = 1, 2, 3, 4)\end{aligned} \tag{3.3}$$

In particular, w_1 and w_2 have negative temporal helicity and satisfy

$$w_1{}^\dagger Aw_1 = w_2{}^\dagger Aw_2 = 1$$

while w_3 and w_4 have positive temporal helicity and satisfy

$$w_3{}^\dagger Aw_3 = w_4{}^\dagger Aw_4 = -1$$

The w_r are given explicitly by

$$w_1 = \tfrac{1}{2}UV\begin{pmatrix}\lambda^{-1}\\0\\-\lambda\\0\\\lambda^{-1}\\0\\\lambda\\0\end{pmatrix}, \qquad w_2 = \tfrac{1}{2}UV\begin{pmatrix}0\\\lambda\\0\\-\lambda^{-}\\0\\\lambda\\0\\\lambda^{-1}\end{pmatrix} \tag{3.4a}$$

$$w_3 = \tfrac{1}{2}UV\begin{pmatrix}\lambda^{-1}\\0\\\lambda\\0\\-\lambda^{-1}\\0\\\lambda\\0\end{pmatrix}, \qquad w_4 = \tfrac{1}{2}UV\begin{pmatrix}0\\\lambda\\0\\\lambda^{-1}\\0\\-\lambda\\0\\\lambda^{-1}\end{pmatrix} \tag{3.4b}$$

where

$$\lambda = \left(\frac{E+m}{2m}\right)^{1/2} + \left(\frac{E-m}{2m}\right)^{1/2}$$

so that

$$\lambda^{-1} = \left(\frac{E+m}{2m}\right)^{1/2} - \left(\frac{E-m}{2m}\right)^{1/2}$$

$$\lambda^2 + \lambda^{-2} = 2E/m, \qquad \lambda^2 - \lambda^{-2} = 2|\mathbf{p}|/m$$

and the unitary 8×8 matrices U and V are defined by

$$U \equiv \exp\left[-\frac{i\theta}{2}(-\Sigma_1 \sin\phi + \Sigma_2 \cos\phi)\right]$$

$$V \equiv \exp\left[-\frac{i\Theta}{2}(-\tau_1 \sin\Phi + \tau_2 \cos\Phi)\right]$$

where (θ, ϕ) are the polar angles of $\mathbf{p}$ and (Θ, Φ) those of $\mathbf{E}$. The properties (3.1) and (3.3) of the w_r are easily verified in the special case

$$P^\mu = (0, 0, |\mathbf{p}|, 0, 0, E)$$

corresponding to $\theta = \Theta = 0$, $U = V = I_8$, and their truth for general P^μ immediately follows from the relations

$$(\gamma^\mu P_\mu) UV = UV(E\gamma^6 - |\mathbf{p}|\gamma^3)$$

$$(\mathbf{\Sigma} \cdot \hat{\mathbf{p}}) UV = UV\Sigma_3, \qquad (\boldsymbol{\tau} \cdot \hat{\mathbf{E}}) UV = UV\tau_3$$

which express the facts that U represents a spatial rotation sending (0, 0, 1) to $\hat{\mathbf{p}}$ and V a temporal rotation sending (0, 0, 1) to $\hat{\mathbf{E}}$.

4. CHARGE CONJUGATION

In the presence of an electromagnetic field with 6-potential $\mathscr{A}_\mu$, the Dirac equation (1.1) for a particle of mass m and charge e becomes

$$\left[\gamma^\mu\left(i\frac{\partial}{\partial x^\mu} - e\mathscr{A}_\mu\right) - m\right]\psi = 0 \tag{4.1}$$

The operation $\psi \to \psi_c$ of charge conjugation must reverse the sign of the charge, so ψ_c must satisfy

$$\left[\gamma^\mu\left(i\frac{\partial}{\partial x^\mu} + e\mathscr{A}_\mu\right) - m\right]\psi_c = 0 \tag{4.2}$$

Taking the complex conjugate of (4.1) and assuming that $\psi_c = C\psi^*$ for some constant 8×8 matrix C, we obtain

$$\left[-\gamma^{\mu *}\left(i\frac{\partial}{\partial x^\mu} + e\mathscr{A}_\mu\right) - m\right] C^{-1}\psi_c = 0$$

which is equivalent to (4.2) provided that

$$-\gamma^{\mu *} C^{-1} = C^{-1}\gamma^\mu$$

that is, provided

$$C^{-1}\gamma^\mu C = -\gamma^{\mu *}$$

Since γ^1, γ^3, γ^4, and γ^6 are real, and γ^2 and γ^5 purely imaginary, this condition may be satisfied by taking

$$C = \gamma^1\gamma^3\gamma^4\gamma^6 = \begin{pmatrix} 0 & 0 & 0 & i\sigma_2 \\ 0 & 0 & -i\sigma_2 & 0 \\ 0 & i\sigma_2 & 0 & 0 \\ -i\sigma_2 & 0 & 0 & 0 \end{pmatrix}$$

which also has the properties

$$C^2 = I_8, \qquad C^\dagger = C, \qquad AC = CA$$

From these we immediately deduce that

$$\psi_c{}^\dagger A \psi_c = \psi^\dagger A \psi \qquad \text{and} \qquad \psi_c{}^\dagger A \gamma^\mu \psi_c = -\psi^\dagger A \gamma^\mu \psi$$

for any ψ. Since

$$(\gamma^\mu P_\mu)C = -C(\gamma^\mu P_\mu)^*$$

it is clear from (3.1) that charge conjugation must reverse the sign of the 6-momentum of a plane wave solution of the free Dirac equation (1.1). Since

$$\mathbf{\Sigma} C = -C\mathbf{\Sigma}^* \qquad \text{and} \qquad \boldsymbol{\tau} C = -C\boldsymbol{\tau}^*$$

we see that spatial and temporal helicities will be invariant under charge conjugation. In fact, the plane wave solutions w_r defined by (3.4) satisfy

$$w_{rc}(p) = Cw_r{}^*(p) = -i\exp\{i[(-1)^r\phi - (-1)^{r(r+1)/2}\Phi]\} w_r(-p)$$

as is easily proved using the evident relations

$$CU^* = UC \qquad \text{and} \qquad CV^* = VC$$

5. CONCLUSIONS

The results obtained are closely analogous to corresponding results of the ordinary four-dimensional theory. However, there is the important difference in the six-dimensional case that the mass hyperboloid $P^{\mu}P_{\mu}=m^2$ is connected, unlike its two-sheeted counterpart in four dimensions. Because of this, there is no dichotomy in the six-dimensional theory between plane wave solutions of positive and negative energy. On the other hand, both theories possess plane wave solutions w for which $w^{\dagger}Aw$ (or its four-dimensional analogue) has either sign. In the four-dimensional theory, the solutions for which this quantity is positive are precisely the positive-energy solutions. In the six-dimensional theory, it follows from (3.3) that the solutions with $w^{\dagger}Aw>0$ are the ones which have negative temporal helicity. There is therefore a sense in which positive energy in the four-dimensional case corresponds to negative temporal helicity in the six-dimensional case. It is tempting to conclude that states of negative and positive temporal helicity in the six-dimensional theory correspond to particles and antiparticles, respectively. However, this interpretation is untenable, since temporal helicity is invariant under charge conjugation.

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